Exponential Functions

Laws of exponents : a, b, c fixed numbers, $a \neq 0$

1/ $a^1 = a$

2/ $a^0 = 1$

3/ $a^{b+c} = a^b \cdot a^c$

4/ $a^{(bc)} = (a^b)^c$

5/ $(ab)^c = a^c b^c$

6/ $a^{-b} = \frac{1}{a^b}$

Super Important!

Warning: $(a+b)^c \neq a^c + b^c$

very common error

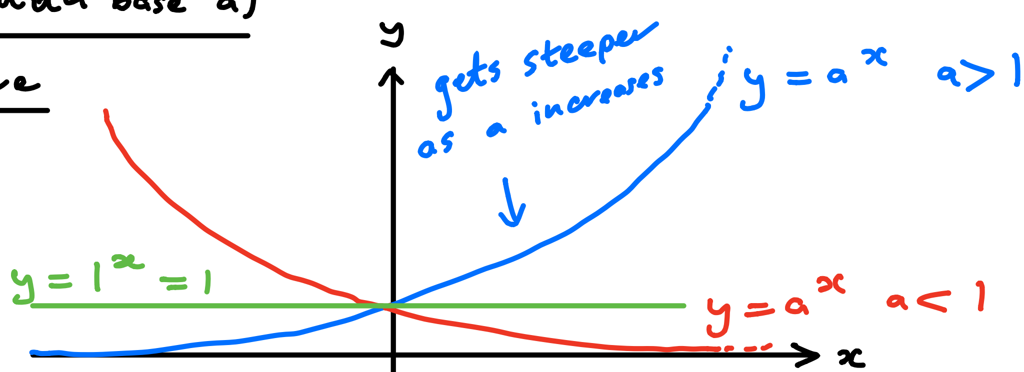
$(1+1)^2 = 2^2 = 4$

$1^2 + 1^2 = 1 + 1 = 2$ ($\neq$)

Fix $a > 0$.

Exponential Function (with base a) $= a^x$

Picture



Domain = all real numbers

Range = $(0, \infty)$ $(a \neq 1)$

Compound Interest

P = amount in savings account at $t = 0$ (time)

r = annual interest rate $(0 \leq r \leq 1)$

$f(t)$ = balance at time t (in years)

$f(0) = P$ interest compoundent at end of year

$f(1) = P + rP = P(1+r)$

$f(2) = (P+rP) + r(P+rP) = P(1+r)^2$

$f(t) = P(1+r)^t$ (t whole number)

Q: What happens if we compound interest more frequently.

Lets compound m times in a year:

annual interest rate divided by m.

$f(0) = P$

$f(1) = P\left(1+\frac{r}{m}\right)^m$

$f(2) = P\left(1+\frac{r}{m}\right)^{2m}$

$f(t) = P\left(1+\frac{r}{m}\right)^{tm}$ (t whole number)

<u>Important Example</u>: $P = 1$, $r = 1$

$\underline{m=1} \Rightarrow f(1) = 1\cdot(1+1)^1 = 2$

$\underline{m=2} \Rightarrow f(1) = 1\cdot\left(1+\frac{1}{2}\right)^2 = 2.25$

$\underline{m=3} \Rightarrow f(1) = 1\cdot\left(1+\frac{1}{3}\right)^3 = 2.37$

$\underline{m=8} \Rightarrow f(1) = 1\cdot\left(1+\frac{1}{8}\right)^8 = 2.5658$

$\underline{m=1000} \Rightarrow f(1) = 1\cdot\left(1+\frac{1}{1000}\right)^{1000} = 2.7169$

$m=100000 \Rightarrow f(1) = 1\cdot\left(1+\frac{1}{100000}\right)^{100000} = 2.7183$

<u>Definition</u>: As n becomes larger and larger, $\left(1+\frac{1}{n}\right)^n$ gets closer and closer to a fixed constant, which we call e

nasty decimal, like π.

$$e \approx 2.718281828\ldots$$

What about for different r and t?

$$f(t) = P\left(1+\frac{r}{m}\right)^{mt}$$

$$= P\left(1+\frac{1}{(m/r)}\right)^{m/r \cdot rt}$$

$$= P\left(\left(1+\frac{1}{(m/r)}\right)^{m/r}\right)^{rt}$$

As m grows so does m/r hence $\left(1+1/(m/r)\right)^{m/r}$ gets closer and closer to e.

<u>Conclusion</u>: As m increases (ie we compound more frequently) $f(t)$ gets closer and closer to Pe^{rt}.

<u>Continuous Compound Interest</u>: If a deposit of P dollars is invested at annual interest rate r, which is to be compounded continuously, then

$$f(t) = Pe^{rt}$$

Balance at time t in years after initial deposit

Many banks offer accounts where interest is compounded continuously.